RETURN ON AUERBACH THEOREM ABOUT BOUNDED LINEAR GROUPS

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Abstract. This work revisits the theorem given in 1932 by Auerbach about bounded linear groups.

1. Introduction

In 1932 [1], Auerbach proved that in any finite vector space each bounded linear group left invariant a quadratic and positive form. This work revisits his proof with modern notation, and aims at paying attention to weaknesses. Only the real field $\mathbb{R}$ is treated here.

2. Definitions and Theorem

The vector space, noted E, is considered on the real field $\mathbb{R}$ and of finite dimension. Its norm is general, so it can be non-euclidean. So E is a finite Banach space.

Definition 1. A quadratic form is defined as an application $Q : E \to \mathbb{R}$, such that $Q(x) = B(x,x)$, for some bi-linear and symmetric form $B : E \times E \to \mathbb{R}$.

Definition 2. A quadratic form is said "positive" if $\forall x \in E, x \neq 0 \Rightarrow B(x,x) > 0$.

The set of linear applications over E is a vector space $\mathcal{L}(E)$. Also the norm of E induces a norm on $\mathcal{L}(E)$ as follows.

Definition 3. The norm of a linear application $A : E \to E$ is defined as:
$\|A\| \triangleq Inf\{c \in \mathbb{R} : \forall x \in E, \|gx\| \leq c\|x\|\}$

Definition 4. A linear group G is said "bounded" if $Sup\{\|g\| : g \in G\} < \infty$.

Theorem 5. *If G is a bounded linear group then there exists a quadratic and positive form Q that is invariant by G: $\forall g \in G, \forall x \in E, Q(gx) = Q(x)$.*

3. Auxiliary definitions, lemmas, and proof plan

Let us rephrase the proof of [1].

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3.1. **The set of quadratic forms is a vector space $\mathcal{Q}(E)$.** To each quadratic form Q one associates a bi-linear and symmetric form B, such that $Q(x) = B(x,x)$. This association is unique.

Indeed:

$$Q(x+y) = B(x+y, x+y) = B(x,x) + B(y,y) + 2B(x,y) = Q(x) + Q(y) + 2B(x,y)$$

So:

$$B(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$$

It follows that the linear combination of two quadratic forms $Q = \lambda_1 Q_1 + \lambda_2 Q_2$ makes sense because associated to the following bi-linear form $B = \lambda_1 B_1 + \lambda_2 B_2$, indeed:

$$B(x,y) = \lambda_1 B_1(x,y) + \lambda_2 B_2(x,y)$$

$$B(x,y) = \frac{\lambda_1}{2}(Q_1(x+y) - Q_1(x) - Q_1(y)) + \frac{\lambda_2}{2}(Q_2(x+y) - Q_2(x) - Q_2(y))$$

$$B(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$$

The neutral element $Q = 0$ exists, which is associated to the null bi-linear and symmetric form.

3.2. **Let us defined a norm over the vector space $\mathcal{Q}(E)$.** Let us check that $Q \rightarrow Inf\left\{c \in \mathbb{R} : \forall x \in E, |Q(x)| \leq c\|x\|^2\right\}$is a norm, noting that by construction $|Q(x)| \leq \|Q\| \cdot \|x\|^2$.

- $\|Q\| = 0 \Rightarrow \forall x \in E, |Q(x)| \leq 0 \Rightarrow \forall x \in E, Q(x) = 0 \Rightarrow Q = 0$
- $\forall x \in E, |\lambda Q(x)| = |\lambda| \cdot |Q(x)| \Rightarrow \forall x \in E, |\lambda Q(x)| \leq |\lambda| \cdot \|Q\| \cdot \|x\|^2 \Rightarrow \|\lambda Q\| = |\lambda| \cdot \|Q\|$
- $\forall x \in E, |Q_1(x) + Q_2(x)| \leq |Q_1(x)| + |Q_2(x)| \leq (\|Q_1\| + \|Q_2\|) \cdot \|x\|^2 \Rightarrow \|Q_1 + Q_2\| \leq \|Q_1\| + \|Q_2\|$

3.3. **To each linear application $A : E \rightarrow E$ one can associate a linear application $\mathcal{A} : \mathcal{Q}(E) \rightarrow \mathcal{Q}(E)$.** The associated application is defined as $\mathcal{A}(Q)(x) \triangleq Q(Ax)$, and it is linear. Indeed:

$\mathcal{A}(\lambda_1 Q_1 + \lambda_2 Q_2)(x) = (\lambda_1 Q_1 + \lambda_2 Q_2)(Ax) = \lambda_1 Q_1(Ax) + \lambda_2 Q_2(Ax) = (\lambda_1 \mathcal{A}(Q_1) + \lambda_2 \mathcal{A}(Q_2))(x)$.

Let us denotes this mapping $q : \mathcal{L}(E) \rightarrow \mathcal{L}(\mathcal{Q}(E))$, so that one can note $q(A) = \mathcal{A}$.

3.4. **If $A : E \rightarrow E$ is non singular then be also $\mathcal{A} : \mathcal{Q}(E) \rightarrow \mathcal{Q}(E)$.** Since the considered vector space are of finite dimensions, it is sufficient to proof that linear applications are injective in order to proof their are bijective. Or equivalently that their kernel is reduced to the neutral element. So let us search for the kernel of the associated linear application $\mathcal{A} : \mathcal{Q}(E) \rightarrow \mathcal{Q}(E)$. Let us consider R a quadratic form in this kernel. This means $\mathcal{A}(R) = 0$ or $\forall x \in E, R(Ax) = 0$. Since A is bijective this implies: $\forall y \in E, R(y) = 0$. So one concludes that $R = 0$, which proofs the property.

3.5. **To any linear group G over E one associates a linear group H over $\mathcal{Q}(E)$.** The set H is defined as $H \triangleq \{q(g) : g \in G\}$. It has been proved that for any A, $\mathcal{A} = q(A)$ is a linear application. So each member of H is a linear application. Let us check the axioms of a group for H:

- The composition $\mathcal{A}_1\mathcal{A}_2$ of two linear applications over $\mathcal{Q}(E)$ is a linear application over $\mathcal{Q}(E)$, and with $\mathcal{A}_1 = q(g_1)$ and $\mathcal{A}_2 = q(g_2)$, one obtains $\mathcal{A}_1\mathcal{A}_2 = q(g_1g_2)$. So the set H is stable by composition of its elements.
- The identity of G induces the identity of H.
- Each regular application g of G over E induces a regular application $q(g)$ of H over $\mathcal{Q}(E)$. So each element of H has an inverse inside H.

3.6. **The linear group H is bounded.** Indeed:

- For each $h \in H$, there is $g \in G$ such that $h = q(g)$.
- $\|q(g)(Q)\| = Inf\left\{c \in \mathbb{R} : \forall x \in E, |Q(gx)| \leq c\|x\|^2\right\}$.
- But for $x \neq 0$, $Q(gx) = \|gx\|^2 \cdot Q\left(\frac{gx}{\|gx\|}\right) = \|x\|^2 \cdot \left\|g\frac{x}{\|x\|}\right\|^2 \cdot Q\left(\frac{gx}{\|gx\|}\right)$.
- Considering that $\left|Q\left(\frac{gx}{\|gx\|}\right)\right| \leq \|Q\|$ and $\left\|g\frac{x}{\|x\|}\right\| \leq \|g\|$, one gets $|Q(gx)| \leq \|Q\| \cdot \|g\| \cdot \|x\|^2$.
- So $\|q(g)(Q)\| = \|Q\| \cdot \|g\|$.
- So for each $h \in H$, $\|h\| = \|g\|$ with $h = q(g)$.
- Since G is bounded then be also H.

3.7. **The set of positive quadratic forms is convex.** Indeed if Q_1 and Q_2 are two quadratic and positive forms then be also $\lambda_1Q_1 + \lambda_2Q_2$ for any positive reals $\lambda_1 > 0$ and $\lambda_2 > 0$. As well for convex combination where $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

3.8. **Action of H on a given positive quadratic form.** Let us select a quadratic and positive form Q_0, so being not null. And let us consider the set of quadratic forms $H(Q_0) \triangleq \{h(Q_0) : h \in H\} = \{q(g)(Q_0) : g \in G\}$. This set is not empty since it contains at least the action of the identity that leads to the element Q_0.

3.8.1. *All elements of $H(Q_0)$ are positive quadratic forms.* Indeed:
$\forall g \in G, \forall x \in E, q(g)(Q_0)(x) = Q_0(gx) \geq 0$.

3.8.2. *$H(Q_0)$ is a bounded set for the norm of $\mathcal{Q}(E)$.* Indeed:

- $\forall h \in H, \|h(Q_0)\| \leq \|h\| \cdot \|Q_0\|$.
- Since H is bounded then be also $H(Q_0)$.

3.8.3. *Let us defined $\hat{H}(Q_0)$ the convex extension of $H(Q_0)$.* It is defined as follows:
$\hat{H}(Q_0) \triangleq \left\{\sum_{i=1,n} \lambda_iQ_i : Q_i \in H(Q_0), \lambda_i \geq 0, \sum_{i=1,n} \lambda_i = 1\right\}$
Each element of $\hat{H}(Q_0)$ is also definite positive.

3.8.4. *$\hat{H}(Q_0)$is bounded.* Indeed:
$\left\|\sum_{i=1,n} \lambda_iQ_i\right\| \leq \sum_{i=1,n} \lambda_i \|Q_i\| \leq \left(\sum_{i=1,n} \lambda_i\right) Max\{\|Q_i\| : i \in [1,n]\} = Max\{\|Q_i\| : i \in [1,n]\} \leq \|Q_0\| \cdot Max\{\|h\| : h \in H\}$

3.8.5. *$\hat{H}(Q_0)$is an invariant set by H.* Indeed: $\forall h \in H, h\left(\sum_{i=1,n} \lambda_iQ_i\right) = \sum_{i=1,n} \lambda_ih(Q_i) = \sum_{i=1,n} \lambda_iR_i$ with $R_i \in H(Q_0)$.

3.8.6. *If $H(Q_0)$ is a finite set.* Let us construct a special element $\hat{Q}_0$, the gravity center of $\hat{H}(Q_0)$, as $\hat{Q}_0 = \frac{1}{n}\sum_{i=1,n} Q_i$, where $H(Q_0) = \{Q_i : i \in [1,n]\}$.

Then $\hat{Q}_0$ is invariant under H. Indeed: $\forall h \in H, h\left(\hat{Q}_0\right) = \frac{1}{n}\sum_{i=1,n} h(Q_i) = \frac{1}{n}\sum_{i=1,n} Q_i$, since h is bijective.

3.8.7. *If $H(Q_0)$ is a not a finite set.* Also in this general case, Auerbach claimed that it should be possible to define the gravity center $\hat{Q}_0$ of $\hat{H}(Q_0)$ and to proof its invariance under H. However Auerbach did not elaborate on this case.

4. Analysis and comments of the proof

The strategy of the proof is to explicitly construct one positive quadratic form $\hat{Q}_0$ that is left invariant by the bounded linear group G. For that purpose, to the bounded linear group G operating over the normed vector space E, it is associated the bounded linear group H operating over the normed vector space $\mathcal{Q}(E)$. Then one considers the action of the group H on a given definite positive form Q_0. The convex extension $\hat{H}(Q_0)$ of the orbit $H(Q_0)$ is bounded and globally invariant under the bounded linear group H. It is expected to construct a point $\hat{Q}_0$ of $\hat{H}(Q_0)$ that is invariant under H, as the gravity center of $\hat{H}(Q_0)$. This is trivial when the group G is finite, since the group H is also finite, as well as the orbit $H(Q_0)$. But when the group G has infinite number of elements, being bounded does not tell enough for being able to make sense for "integration". This is when the "Haar measure" enters in the game. Or, maybe there is the option of proving the existence of the point $\hat{Q}_0$ without constructing it explicitly? Could a fixed point theorem (applicable to Banach space) work (like Sauder and Tychonoff)? Either "integration" or "continuity" path would require some topology property of the group G, then transported to the group H.

5. License

References

[1] Auerbach. Sur les groupes bornés de substitutions linéaires. *Comptes Rendus de l'Académie des Sciences*, xxx:1367–1369, 1932.

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