HOMEWORK ON JORDAN-NEUMANN INNER PRODUCT SPACE CHARACTERIZATION

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ABSTRACT. This is just a homework on the characterization of inner product spaces by the parallelogram law. In 1935, Jordan an von Neumann published such a theorem, applicable to both real and complex linear spaces.

1. INTRODUCTION

In 1935, Jordan and von Neumann provided [1] an algebraic characterization of inner product space applicable to a vector normed space E. It involves the *parallelogram law*:

$$\forall (x,y) \in E^2, \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \tag{1.1}$$

The theorem states that if 1.1 holds then an inner product $f : E \times E \to \mathbb{R}$ can be defined as:

$$f(x,y) \triangleq \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) \tag{1.2}$$

This article is just a homework that revisits the proof in the sole case of real vector space.

2. PROOF

Let us check that f defined by 1.2 under 1.1 satisfies each axiom on an inner product.

2.1. **Symmetry.** The function f is symmetric in its arguments. This immediately derives from its definition.

2.2. **Positive-definiteness.** The function f is positive.

Indeed, $f(x,x) = \frac{1}{2}\left(\|x+x\|^2 - \|x\|^2 - \|x\|^2\right) = \|x\|^2$. So $f(x,x) \geq 0$, and $f(x,x) = 0 \Rightarrow x = 0$.

2.3. **Linearity.**

2.3.1. *By addition of vectors.* The function f is additive, meaning that $\forall (x,y,z) \in E^3, f(x+y,z) = f(x,z) + f(y,z)$.

Indeed, let us compute $A \triangleq 2\left(f(x+y,z) - f(x,z) - f(y,z)\right)$.

$$A = \left(\left(\|x+y+z\|^2 - \|x+y\|^2 - \|z\|^2\right) - \left(\|x+z\|^2 - \|x\|^2 - \|z\|^2\right) - \left(\|y+z\|^2 - \|y\|^2 - \|z\|^2\right)\right)$$

$$A = \left(\|x+y+z\|^2 + \|z\|^2\right) - \left(\|x+z\|^2 + \|y+z\|^2\right) + \left(\|x\|^2 + \|y\|^2\right) - \|x+y\|^2$$

Let us apply1.1 to the first third groups of terms.

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$$A=\frac{1}{2}\left(\|x+y+2z\|^2+\|x+y\|^2\right)-\frac{1}{2}\left(\|x+y+2z\|^2+\|x-y\|^2\right)+\frac{1}{2}\left(\|x+y\|^2+\|x-y\|^2\right)-\|x+y\|^2$$

Then, one check the cancellation of all terms, so $A=0$.

2.3.2. *By scalar multiplication.* The function f is multiplicative, meaning that $\forall(x,z)\in E^2,\forall\lambda\in\mathbb{R},f(\lambda x,z)=f(x,z)$.

Let us check this property by growing sets of scalars.

- From the definition of f, one immediately verifies that $\forall z\in E,f(0,z)=0$. Combined with the additivity, one checks that $\forall(x,z)\in E^2,f(x-x,z)=0=f(x,z)+f(-x,z)\Rightarrow f(-x,z)=-f(x,z)$.
- From the additivity, one deduces $\forall(x,z)\in E^2,f(2x,z)=2f(x,z)$, and by recurrence $\forall(x,z)\in E^2,\forall n\in\mathbb{N},f(nx,z)=nf(x,z)$.
- So, at this step: $\forall(x,z)\in E^2,\forall n\in\mathbb{Z},f(nx,z)=nf(x,z)$.
- Then remarking that $\forall(x,z)\in E^2,\forall n\in\mathbb{Z},f\left(n\frac{x}{n},z\right)=f(x,z)=nf\left(\frac{x}{n},z\right)\Rightarrow f\left(\frac{x}{n},z\right)=\frac{1}{n}f(x,z)$, one proofs the linearity regarding all rational numbers: $\forall(x,z)\in E^2,\forall r\in\mathbb{Q},f(rx,z)=rf(x,z)$.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, a continuity argument would permit to conclude to the linearity of f regarding the reals.

For any fixed $(x,z)\in E^2$, let us consider the application $g:\mathbb{R}\to\mathbb{R}$, defined by $g(\lambda)\triangleq f(\lambda x,z)$. Let us check that g is continuous. For that, it is sufficient to proof that $h:\lambda\mapsto\|\lambda x+z\|$, for any fixed $(x,z)\in E^2$, is continuous, since g can be expressed as sums and squares that maintain the continuity.

Let us find a majoration of $|h(\lambda)-h(\mu)|$ for any pair or reals (λ,μ).

$$|h(\lambda)-h(\mu)|=|\|\lambda x+z\|-\|\mu x+z\||\leq\|\lambda x-\mu x\|=|\lambda-\mu|\,\|x\|$$

Indeed, for any norm, without using 1.1, one gets:

$$\forall(x,y)\in E^2,|(\|x\|-\|y\|)|\leq\|x-y\|$$

This derives from successive application of triangular inequality $\|x+y\|\leq\|x\|+\|y\|$:

$$\|y\|=\|x+(y-x)\|\leq\|x\|+\|y-x\|\Rightarrow\|y\|-\|x\|\leq\|y-x\|$$

$$\|x\|=\|y+(x-y)\|\leq\|y\|+\|x-y\|\Rightarrow\|x\|-\|y\|\leq\|x-y\|$$

This achieves the proof of linearity of f .

3. License

References

[1] P. Jordan and J. V. Neumann. On inner products in linear, metric spaces. *The Annals of Mathematics*, 36(3):pp. 719–723, 1935.

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