INNER PRODUCT SPACE CHARACTERIZATION BY ISOMETRY GROUP OVER UNITY SPHERE

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Abstract. This work registers a characterization of euclidean space (a.k.a. inner product space) : the existence of an isometry group transporting points of unity sphere.

1. Introduction

As a mathematic amateur, I am looking for deep understanding of the role of "power 2" in the Pythagorean theorem $c^2 = a^2 + b^2$. The answer "it derives from axioms from Euclide" does not sound satisfactory. Indeed, my point is not to find a valid proof, but to find alternate axioms. Let us assume a geometry over the set E considered as 1) a vector space over the real field $\mathbb{R}$; 2) with finite dimensions ; 3) equiped with a norm. Which additional axioms have to be added in order to conclude that E is euclidean, or equivalently, that its norm is deriving from an inner product. Such additionnal axioms are searched in topology and in group formulations, as opposed to calculational ones, like the "polarisation identity". This work, not innovative, and merely an homework, gathers theorems that proof the following axiom solution: if a group G of isometries allows transportation between any pair of points of the unity sphere of E, then E is euclidean.

The section 2 formulates the theorem to be proven, gives definitions and auxilliary theorems. The section 3 provides the proof. The section 4 concludes with comments.

2. Definitions and Theorems

A general vector normed space E is considered on the real field $\mathbb{R}$. Only the case of finite dimension is considered. A priori, its norm is not euclidean. The theorem to be proven is the following:

Theorem 1 (Isometric transport). *If a set G of isometries, over the normed vector space E, can transport any point x of the unity sphere S onto any other $y \in S$, then the norm of E derives from an inner product.*

The transport part of the hypothesis can be written:

$$\forall (x, y) \in S^2, \exists g \in G, y = g(x)$$

The following definitions aim at clarifying involved concepts.

Definition 2 (Isometry). An application $f : E \to F$, from a vector normed space E onto another vector normed space F, is an isometry if :

$$\forall (x, y) \in E^2, \|f(x) - f(y)\| = \|x - y\|$$

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Definition 3 (Quadratic form)**.** A quadratic form is defined as an application $Q : E \to \mathbb{R}$, such that $Q(x) = B(x, x)$, for some bilinear and symmetric form $B : E \times E \to \mathbb{R}$.

Definition 4 (Positive QF)**.** A quadratic form is said "positive" if:

$$\forall x \in E, x \neq 0 \Rightarrow B(x, x) > 0$$

The set of linear applications over E is a vector space $\mathcal{L}(E)$. And the norm of E induces a norm on $\mathcal{L}(E)$ as follows.

Definition 5 (Norm over$\mathcal{L}(E)$)**.** The norm $\|A\|$ of a linear application $A : E \to E$ is defined as:

$$\|A\| \triangleq Inf \{c \in \mathbb{R} : \forall x \in E, \|gx\| \leq c \|x\|\}$$

Definition 6 (Bounded group)**.** A linear group G is said "bounded" if:

$$Sup \{\|g\| : g \in G\} < \infty$$

The following auxialliary theorems are used in the proof.

Theorem 7 (Mazur Ulam [2])**.** *An isometry $f : E \to F$, from a vector normed space E onto another vector normed space F, with $f(0) = 0$, is always a linear application.*

Theorem 8 (Auerbach [1])**.** *If G is a bounded linear group then there exists a quadratic and positive form Q that is invariant by G:*

$$\forall g \in G, \forall x \in E, Q(gx) = Q(x)$$

3. Proof

The theorem 1 is proven according to the following plan:

- The theorem 7 implies than any isometry is linear. So the set G is necessarly composed of linear isometries.
- Any isometry is bijective. Indeed, $f(x) = f(y) \Rightarrow \|f(x) - f(y)\| = 0 = \|x - y\| \Rightarrow x = y$, which means f is injective. Since, E has finite dimension, then f is also surjective.
- If f and g are isometries then be also $f \circ g$. Indeed: $\|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\| = \|x - y\|$.
- So the set of isometries over E is a linear group.
- The group of isometries over E is bounded. Indeed, for any isometry g over E :
$$\forall x \in E, \|gx\| = \|x\| \Rightarrow \|g\| = 1 < \infty$$
- Any isometry is linear, it maps the unity sphere onto itself. So does the composition of any two linear isometries. So G is a subgroup of the isometries group over E.
- The theorem 8 implies there is some positive quadratic form Q, which is left invariant by the group G.

- It is trivial to deduce that the quadratic form Q is constant over the unity sphere. Indeed:
  - The theorem 8 means: $\forall x \in E, \forall g \in G, Q(x) = Q(g(x))$
  - So in particular for the unity sphere: $\forall x \in S, \forall g \in G, Q(x) = Q(g(x))$
  - The transportation hypothesis allows to connect any two points of the sphere:
  $$\forall x \in S, \forall y \in S, \exists g \in G, y = g(x)$$
  - This allows the conclusion: $\forall x \in S, \forall y \in S, Q(y) = Q(x)$
- The conclusion is obtained by tranlsating the previous result in term of the, a priori, general norm $\|\cdot\|$ over E:
  - $\exists c \in \mathbb{R}, c > 0, \forall u \in E, \|u\| = 1 \Rightarrow Q(u) = c$
  - Then equivalently: $\exists c \in \mathbb{R}, c > 0, \forall x \in E, x \neq 0 \Rightarrow Q\left(\frac{x}{\|x\|}\right) = c$
  - Or noting $B(x, y)$, the binlinear and symetric form associated to $Q(x)$:
  - $\exists c \in \mathbb{R}, c > 0, \forall x \in E, x \neq 0 \Rightarrow B\left(\frac{x}{\|x\|}, \frac{x}{\|x\|}\right) = c = \frac{B(x,x)}{\|x\|^2}$
  - So one concludes: $\exists c \in \mathbb{R}, c > 0, \forall x \in E, \|x\| = \sqrt{\frac{B(x,x)}{c}}$, which means that the norm derives from an inner product.

4. Conclusion

The "power two" of Pythagorean theorem, with this perspective, emerges from requested kind of "isotropy" and "homegenity" properties.

The two main theorems [2]and [1] are worked on http://lucas.borboleta.blog.free.fr. It should be noted that [1] is not conclusive, because a more elaborated mesaure theory is needed ("Haar measure"?) in order to define the "gravity center" of the orbit of a quadratic form under action of a bounded linear group.

5. License

References

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