LINEAR ISOMETRIES IN $\mathbb{R}^2$ FOR P-NORMS

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Abstract. This article determines the set linear isometries in $\mathbb{R}^2$ for any p-norm $\|(x,y)\|_p \triangleq (|x|^p + |y|^p)^{\frac{1}{p}}$, with $p \geq 1$, and concludes that this set, composing a group, is discrete and finite for any p, excepted for $p = 2$, that is for the Euclidean plan. This illustrates the very special properties of the Euclidean space (by plan to space extension) among normed spaces.

1. Introduction

1.1. **Motivation.** The Euclidean space, when defined by a norm derived from a scalar product, provides the students with algebra for deriving geometrical properties. This is fine. However, for a deeper understanding, there is the need to, also, know about equivalent axioms, more geometrical than algebraical. This article does not provides such equivalent axioms, but contributes to this goal by illustrating some properties about the linear isometries in $\mathbb{R}^2$ for any p-norm $\|(x,y)\|_p \triangleq (|x|^p + |y|^p)^{\frac{1}{p}}$, with $p \geq 1$: except for $p = 2$, the group of linear isometries is discrete and finite .

Let us consider this article as a shared homework since professional mathematicians have most probably worked out this matter.

1.2. **Article overview.** The first section, establishes that, for any norm, the set of linear isometries in $\mathbb{R}^n$ is composing a group.

Then, the next sections focus on any linear isometry in $\mathbb{R}^2$, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for a norm $\|\cdot\|_p$, and treat subsequently the following aspects:

- For any norm $\|\cdot\|_p$, the linear transformations $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$, with $(\alpha, \beta, \gamma, \delta) \in \{-1, +1\}^4$, are composing a discrete and finite sub-group.
- For any norm $\|\cdot\|_p$, a linear isometry T with $a \cdot b \cdot c \cdot d \neq 0$, so outside the finite sub-group, is possible only for $p = 2$, and requires the form $T = \begin{pmatrix} \alpha\,|a| & \beta\sqrt{1-a^2} \\ \gamma\sqrt{1-a^2} & \delta\,|a| \end{pmatrix}$, with $0 < |a| < 1$ and $\alpha\beta\gamma\delta = -1$.
- For $p = 2$, the required form for T is actually an isometry.

The conclusion envisions an analysis continuation.

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2. The linear isometries are composing a group

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry for the norm $\|\cdot\|$ if $\forall \mathbf{u} \in \mathbb{R}^n, \|T(\mathbf{u})\| = \|\mathbf{u}\|$.

Let us check that the set of linear isometries is composing a group:

- Composing two linear isometries T and S leads to a linear isometry. Indeed, $T \circ S$ is linear, and $\forall \mathbf{u} \in \mathbb{R}^n$, $\|T \circ S(\mathbf{u})\| = \|T(S(\mathbf{u}))\| = \|S(\mathbf{u})\| = \|\mathbf{u}\|$, so the norm is unchanged by $T \circ S$.
- The identity transformation is linear, and obviously let invariant the norm.
- A linear isometry T is always bijective, because its kernel is reduced to the null vector. Indeed, $T(\mathbf{u}) = \mathbf{0} \Rightarrow \|T(\mathbf{u})\| = \mathbf{0}$ but, $\|T(\mathbf{u})\| = \|\mathbf{u}\|$, so $\|\mathbf{u}\| = 0 \Rightarrow \mathbf{u} = \mathbf{0}$. The inverse T^{-1} is also an isometry. Indeed, $T(\mathbf{v}) = \mathbf{u} \Leftrightarrow \mathbf{T^{-1}}(\mathbf{u}) = \mathbf{v}$, so $\|T(\mathbf{v})\| = \|\mathbf{u}\|$ and $\|T^{-1}(\mathbf{u})\| = \|\mathbf{v}\|$. Since T is an isometry then$\|T(\mathbf{v})\| = \|\mathbf{v}\|$, and finally $\|T^{-1}(\mathbf{u})\| = \|\mathbf{u}\|$.

3. The discrete and finite sub-group of linear isometries

Let us check that, in $\mathbb{R}^2$, the set of linear transformations $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$, with $(\alpha, \beta, \gamma, \delta) \in \{-1, +1\}^4$, are composing a discrete (obviously finite) group for any norm $\|\cdot\|_p$. Geometrically, this group is generated by the median and diagonal reflections.

- The two forms of linear transformations let invariant the norm:
  - $\left\| \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p = (|\alpha x|^p + |\delta y|^p)^{\frac{1}{p}} = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p$
  - $\left\| \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p = (|\beta y|^p + |\gamma x|^p)^{\frac{1}{p}} = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p$
- The two forms of linear transformations are stable by compositions:
  - $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 & 0 \\ 0 & \delta_1\delta_2 \end{pmatrix}$
  - $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1\beta_2 \\ \delta_1\gamma_2 & 0 \end{pmatrix}$
  - $\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1\gamma_2 & 0 \\ 0 & \gamma_1\beta_2 \end{pmatrix}$
  - $\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1\delta_2 \\ \gamma_1\alpha_2 & 0 \end{pmatrix}$

4. Constraints for the group of linear isometries

For any norm $\|\cdot\|_p$ in $\mathbb{R}^2$, let us consider a linear isometry $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In general,$\forall (x, y) \in \mathbb{R}^2, \|T(x, y)\|_p = \|(x, y)\|_p$, and in particular, $\|T(1, 0)\|_p = 1$ and $\|T(0, 1)\|_p = 1$ lead to the following constraints:

$$|a|^p + |c|^p = 1 \tag{4.1}$$

$$|b|^p + |d|^p = 1 \tag{4.2}$$

If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear isometry, then its inverse $T^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is also a linear isometry. So $\|T^{-1}(1, 0)\|_p = 1$ and $\|T^{-1}(0, 1)\|_p = 1$ lead to the following constraints:

$$|a|^p + |b|^p = |ad - bc|^p \tag{4.3}$$

$$|c|^p + |d|^p = |ad - bc|^p \tag{4.4}$$

Summing equations 4.3 and 4.4, and using 4.1 and 4.2leads to:

$$2\,|ad - bc|^p = |c|^p + |d|^p + |a|^p + |b|^p = 2 \tag{4.5}$$

So the constraints derived from T^{-1} can be rewritten as:

$$|a|^p + |b|^p = 1 \tag{4.6}$$

$$|c|^p + |d|^p = 1 \tag{4.7}$$

$$|ad - bc| = 1 \tag{4.8}$$

Equations 4.1 and 4.6 implies $|c| = |b|$. Equations 4.1 and 4.7 implies $|d| = |a|$.

Let us express $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha\,|a| & \beta\,|b| \\ \gamma\,|b| & \delta\,|a| \end{pmatrix}$ with $(\alpha, \beta, \gamma, \delta) \in \{-1, +1\}^4$. Using this information in 4.8 leads to:

$$\left|\alpha\delta a^2 - \beta\gamma b^2\right| = 1 \tag{4.9}$$

Let us study the particular cases where $a \cdot b = 0$:

- Case $b = 0$: Equation 4.9 implies $|a| = 1$, and yields to a linear isometry $T = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$.
- Case $a = 0$: Equation 4.9 implies $|b| = 1$, and yields to a linear isometry $T = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$.

These particular linear isometries, where $a \cdot b = 0$, are composing the discrete group.

Let us continue with the general cases where $a \cdot b \neq 0$. Equation 4.9 implies:

$$\left|b^2 - \alpha\beta\delta\gamma a^2\right| = 1 \tag{4.10}$$

With $\varepsilon \in \{-1, +1\}$, this is equivalent to:

$$b^2 - \alpha\beta\delta\gamma a^2 = \varepsilon \Leftrightarrow b^2 = \varepsilon + \alpha\beta\delta\gamma a^2 \tag{4.11}$$

Let us analyze the four possibilities for $(\varepsilon, \alpha\beta\gamma\delta)$, knowing that 4.6 and $a \cdot b \neq 0$ requires $0 < |a| < 1$ and $0 < |b| < 1$:

- $(\varepsilon, \alpha\beta\gamma\delta) = (-1, -1)$: this leads to $b^2 = -1 - a^2$, which is never possible.
- $(\varepsilon, \alpha\beta\gamma\delta) = (-1, +1)$: this leads to $b^2 = -1 + a^2$, which is possible only if $|a| = 1$ and $b = 0$, but this is excluded by $a \cdot b \neq 0$ hypothesis.
- $(\varepsilon, \alpha\beta\gamma\delta) = (+1, +1)$: this leads to $b^2 = 1 + a^2$, which is possible only if $a = 0$ and $|b| = 1$, but this is excluded by $a \cdot b \neq 0$ hypothesis .
- $(\varepsilon, \alpha\beta\gamma\delta) = (+1, -1)$: this leads to $b^2 = 1 - a^2$, which is possible, for any $0 < |a| < 1$, since it also implies$0 < |b| < 1$.

Therefore, let us continue with the analysis with the unique possibility for $(\varepsilon, \alpha\beta\delta)$:

$$b^2 = 1 - a^2 \tag{4.12}$$

$$\alpha\beta\gamma\delta = -1 \tag{4.13}$$

The equation 4.6 can be rewritten:

$$\left(a^2\right)^{\frac{p}{2}} + \left(1 - a^2\right)^{\frac{p}{2}} = 1 \tag{4.14}$$

For $p = 2$, any value $a^2 \in \,]0, 1[$ is solution of equation 4.14, and yields to a linear transformation $T = \begin{pmatrix} \alpha\,|a| & \beta\sqrt{1-a^2} \\ \gamma\sqrt{1-a^2} & \delta\,|a| \end{pmatrix}$ with $\alpha\beta\gamma\delta = -1$.

For $p \neq 2$, let us study the function $x \in [0, 1] \mapsto f(x) = x^{\frac{p}{2}} + (1 - x)^{\frac{p}{2}} - 1$, for its values and its first and second derivatives:

- $f(0) = f(1) = 0$, $f\left(\frac{1}{2}\right) = 2^{\frac{2-p}{2}} - 1$
- $f^{(1)}(x) = \frac{p}{2}\left(x^{\frac{p}{2}-1} - (1-x)^{\frac{p}{2}-1}\right)$, $f^{(1)}(x) = 0 \Leftrightarrow x = \frac{1}{2}$
- $f^{(2)}(x) = \frac{p}{2}\left(\frac{p}{2} - 1\right)\left(x^{\frac{p}{2}-2} + (1-x)^{\frac{p}{2}-2}\right) \neq 0$
- For $p > 2$:
  - $f^{(2)}(x) > 0$
  - $f^{(1)}(x) < 0$ in $\left]0, \frac{1}{2}\right[$, and $f^{(1)}(x) > 0$ in $\left]\frac{1}{2}, 1\right[$
  - $f(x) < 0$ in $]0, 1[$
- For $p < 2$:
  - $f^{(2)}(x) < 0$
  - $f^{(1)}(x) > 0$ in $\left]0, \frac{1}{2}\right[$, and $f^{(1)}(x) < 0$ in $\left]\frac{1}{2}, 1\right[$
  - $f(x) > 0$ in $]0, 1[$

Conclusion, for $p \neq 2$, the equation 4.14 has no solution in $]0, 1[$, so the set of isometries is reduced the discrete group.

5. The constraints satisfy the requirement

Let us check that for $p = 2$, any linear transformation $T = \begin{pmatrix} \alpha\,|a| & \beta\sqrt{1-a^2} \\ \gamma\sqrt{1-a^2} & \delta\,|a| \end{pmatrix}$, with $0 < |a| < 1$, $(\alpha, \beta, \gamma, \delta) \in \{-1, +1\}^4$, and $\alpha\beta\gamma\delta = -1$, is an isometry of $\|\cdot\|_2$.

Being an isometry requires from T that $\forall (x, y) \in \mathbb{R}^2, \|T(x, y)\|_2 = \|(x, y)\|_2$.

The structure of T translates the requirement to:

$$\forall (x, y) \in \mathbb{R}^2, \left(\alpha\,|a|\,x + \beta\sqrt{1-a^2}y\right)^2 + \left(\gamma\sqrt{1-a^2}x + \delta\,|a|\,y\right)^2 = x^2 + y^2 \tag{5.1}$$

The constraints $(\alpha, \beta, \gamma, \delta) \in \{-1, +1\}^4$ reduce the requirement to:

$$\forall (x, y) \in \mathbb{R}^2, (\alpha\beta + \gamma\delta)\,|a|\,\sqrt{1-a^2}xy = 0 \tag{5.2}$$

Finally, the constraint $\alpha\beta\gamma\delta = -1 \Leftrightarrow \alpha\beta = -\gamma\delta$ ensures that the requirement is satisfied.

6. Conclusion

This article provides a characteristic property of the euclidean norm among all p-norms. It would be interesting to pursue such analysis by some converse implication, applicable to all norms of $\mathbb{R}^n$, not just limited to p-norms.

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